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C H A P T E R

7 Trigonometric Equations

Recall that trigonometric equations were introduced in section 5–6. In this chapter we first study trigonometric identities; these are very important in the study of the calculus and certain engineering applications. We then examine conditional trigonometric equations in more depth than we did previously.

7–1 Basic trigonometric identities

A computer program is being written that must calculate $\dfrac{1}{\cot \theta + \tan \theta}$ for a varying value θ, which is input into the computer 100 times per second. A more efficient way to compute this expression is desired. Show that calculating the simpler expression $\sin \theta \cos \theta$ will give the same results.

In this section we examine identities. This problem is equivalent to showing that

$$\frac{1}{\cot \theta + \tan \theta} = \sin \theta \cos \theta$$

is an identity. Identities are used to simplify computations in many situations.

Recall from section 5–6 that an **identity** is an equation that is true for every allowed value of its variable (or variables). We have seen the following identities in that and other sections.

Reciprocal identities

$$\csc \theta = \frac{1}{\sin \theta}, \qquad \sec \theta = \frac{1}{\cos \theta}, \qquad \cot \theta = \frac{1}{\tan \theta}$$

$$\sin \theta = \frac{1}{\csc \theta}, \qquad \cos \theta = \frac{1}{\sec \theta}, \qquad \tan \theta = \frac{1}{\cot \theta}$$

Tangent and cotangent identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

The Pythagorean identities

Recall that $\sin^2\theta + \cos^2\theta = 1$ is the fundamental identity. Two other forms of the fundamental identity are

$$\sin^2\theta = 1 - \cos^2\theta \quad \text{and} \quad \cos^2\theta = 1 - \sin^2\theta.$$

If each term in the fundamental identity is divided by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Similarly, if each term of the fundamental identity is divided by $\sin^2\theta$ we obtain the identity $\cot^2\theta + 1 = \csc^2\theta$. These two identities, along with the fundamental identity, are called the **Pythagorean identities.** They are summarized here.

Pythagorean identities	Useful forms	
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\sec^2\theta = \tan^2\theta + 1$	$\tan^2\theta = \sec^2\theta - 1$	$\sec^2\theta - \tan^2\theta = 1$
$\csc^2\theta = \cot^2\theta + 1$	$\cot^2\theta = \csc^2\theta - 1$	$\csc^2\theta - \cot^2\theta = 1$

Example 7–1 A illustrates applications of these identities.

■ ***Example 7–1 A***

Simplify each expression into one term.

1. $1 - \cos^2 4\alpha$

$\sin^2 4\alpha$ — $\sin^2 4\alpha + \cos^2 4\alpha = 1$, so $1 - \cos^2 4\alpha = \sin^2 4\alpha$

2. $(1 - \sec x)(1 + \sec x)$

$1 + \sec x - \sec x - \sec^2 x$ — $(a - b)(a + b) = a^2 + ab - ab - b^2$

$1 - \sec^2 x$ — Collect like terms

$-(\sec^2 x - 1)$ — $a - b = -(b - a)$

$-\tan^2 x$ ■

Verifying identities

One use of identities is in simplifying and transforming trigonometric expressions. This is illustrated in example 7–1 B. We proceed by replacing given parts of an expression by equivalent parts from the identities just summarized. *There are many correct sequence of steps!* We proceed by trial and error, guided by past experience. If we can show that one member of an equation can be transformed into the other member by replacing expressions using identities and performing algebraic transformations, then we say we have *verified* the identity.

■ ***Example 7–1 B*** Verify that each equation is an identity by showing that the left member of the identity is equivalent to the right member.

1. $\tan\theta\csc\theta = \sec\theta$

$\tan\theta\csc\theta$

$\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}$ $\qquad \tan\theta = \frac{\sin\theta}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}$

$\frac{1}{\cos\theta}$ $\qquad$ Reduce by a factor of $\sin\theta$

$\sec\theta$ $\qquad \sec\theta = \frac{1}{\cos\theta}$

2. $\frac{1}{\sin\beta - \csc\beta} = -\tan\beta\sec\beta$

$\frac{1}{\sin\beta - \csc\beta}$

$\frac{1}{\sin\beta - \frac{1}{\sin\beta}}$ $\qquad \csc\beta = \frac{1}{\sin\beta}$

$\frac{1}{\sin\beta - \frac{1}{\sin\beta}}\cdot\frac{\sin\beta}{\sin\beta}$ $\qquad$ Multiply numerator and denominator by $\sin\beta$

$\frac{\sin\beta}{\sin^2\beta - 1}$ $\qquad \sin\beta\left(\sin\beta - \frac{1}{\sin\beta}\right) = \sin^2\beta - 1$

$\frac{\sin\beta}{-\cos^2\beta}$ $\qquad \cos^2\beta = 1 - \sin^2\beta$, so $-\cos^2\beta = \sin^2\beta - 1$

$-\frac{\sin\beta}{\cos\beta}\cdot\frac{1}{\cos\beta}$

$-\tan\beta\sec\beta$

3. $\frac{\cos^2\alpha}{1 + \sin\alpha} = 1 - \sin\alpha$

$\frac{\cos^2\alpha}{1 + \sin\alpha}$

$\frac{1 - \sin^2\alpha}{1 + \sin\alpha}$ $\qquad \cos^2\theta = 1 - \sin^2\theta$

$\frac{(1 - \sin\alpha)(1 + \sin\alpha)}{1 + \sin\alpha}$ $\qquad m^2 - n^2 = (m - n)(m + n)$

$1 - \sin\alpha$ ■

It is not necessary to transform just one member or the other of an identity. Sometimes it is easier to transform both members, as in example 7–1 C.

■ *Example 7–1 C*

Verify the identity $\sin^2\theta\tan^2\theta + 1 = \sec^2\theta - \cos^2\theta\sec^2\theta + \cos^2\theta$

$\sin^2\theta\tan^2\theta + 1$	$\sec^2\theta - \cos^2\theta\sec^2\theta + \cos^2\theta$
$\sin^2\theta\tan^2\theta + \sin^2\theta + \cos^2\theta$	$\sec^2\theta(1 - \cos^2\theta) + \cos^2\theta$
$\sin^2\theta(\tan^2\theta + 1) + \cos^2\theta$	$\sec^2\theta\sin^2\theta + \cos^2\theta$
$\sin^2\theta\sec^2\theta + \cos^2\theta$	$\dfrac{1}{\cos^2\theta}\sin^2\theta + \cos^2\theta$
$\sin^2\theta\dfrac{1}{\cos^2\theta} + \cos^2\theta$	$\tan^2\theta + \cos^2\theta$
$\tan^2\theta + \cos^2\theta$	

Since the left member and right member can be transformed into the same expression, they are equivalent. ■

Showing an equation is not an identity

Most equations are not identities. To show that this is the case we need to find a value for the variable for which the statement is not true. This value is called a **counter example;** it shows that the equation is *not* an identity.

■ *Example 7–1 D*

Show by counter example that $\cos x + \sin x\cot x = 1$ is not an identity.

Choose a value for which each expression is defined; $x = \dfrac{\pi}{4}$ is such a value.

$$\cos x + \sin x\cot x$$
$$\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cot\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot 1 = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Since $\sqrt{2} \neq 1$ we have shown that the given equation is not an identity.

Note There are usually many counter examples for a given equation. In this example almost any other value would have worked as well. (However, $\dfrac{\pi}{3}$ would not work, because the equation is true for this value.) ■

Mastery points

Can you

- State the reciprocal identities, fundamental identity, and the remaining Pythagorean identities from memory?
- Recognize useful forms of the Pythagorean identities?
- Transform forms of the Pythagorean identities into simpler forms?
- Transform one member of an identity into the other member?
- Show that an equation is not an identity by a counter example?

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Exercise 7–1

Each of the following expressions can be simplified into the form 1, −1, $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$, $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$, $\cot^2\theta$, $\sec^2\theta$, or $\csc^2\theta$. Simplify each expression into one of these forms.

1. $\dfrac{\sin\theta}{\tan\theta}$ **2.** $\dfrac{\cos\theta}{\cot\theta}$ **3.** $\cot\theta\sec\theta$ **4.** $\sec\theta\cot\theta$

5. $\cot^2\theta\sin^2\theta$ **6.** $\sin^2\theta\sec^2\theta$ **7.** $(\tan^2\theta+1)(1-\sin^2\theta)$ **8.** $(1-\cos^2\theta)(1+\cot^2\theta)$

9. $\dfrac{(\sec\theta-1)(\sec\theta+1)}{\sin^2\theta}$ **10.** $\dfrac{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\cos^2\theta}$ **11.** $(\csc x+\cot x)(1-\cos x)$

12. $\cos x+\sin x\tan x$ **13.** $\sec x-\tan x\sin x$ **14.** $(\csc x+1)(\sec x-\tan x)$

15. $\cot x\sec x$ **16.** $\dfrac{\csc x+\sec x}{\tan x+1}$ **17.** $\dfrac{\csc^2\theta-1}{\csc^2\theta}$

18. $\dfrac{\sec^4 y-\tan^4 y}{\sec^2 y+\tan^2 y}$ (*Hint:* Factor $\sec^4 y-\tan^4 y$.) **19.** $\sin x+\cos x\cot x$ **20.** $\tan x\csc x\cos x$

21. $\dfrac{\csc\theta\sin\theta}{\cot\theta}$ **22.** $\tan^2\theta(\cot^2\theta+1)$ **23.** $\dfrac{\tan\theta\cot\theta}{\sin\theta}$ **24.** $\cos\theta(\sec\theta-\cos\theta)$

25. $\cos^2\theta(1+\cot^2\theta)$ **26.** $\csc^2\theta(1-\cos^2\theta)$ **27.** $\sin^2\theta(\csc^2\theta-1)$ **28.** $\cot^2\theta-\csc^2\theta$

29. $\tan^2\theta-\sec^2\theta$ **30.** $\sec^2\theta(\csc^2\theta-1)$ **31.** $\dfrac{\cot\theta\sec\theta}{\csc\theta}$ **32.** $\dfrac{\sec\theta}{\tan\theta\csc\theta}$

Verify the following identities.

33. $\csc\theta+\cot\theta=\dfrac{1+\cos\theta}{\sin\theta}$ **34.** $\tan\theta+\sec\theta=\dfrac{1+\sin\theta}{\cos\theta}$ **35.** $\dfrac{\csc\theta}{\sec\theta+\tan\theta}=\dfrac{\cos\theta}{\sin\theta+\sin^2\theta}$

36. $\dfrac{\sec\theta}{\csc\theta+\cot\theta}=\dfrac{\tan\theta}{1+\cos\theta}$ **37.** $\dfrac{1+\csc\theta}{1+\sec\theta}=\cot\theta\left(\dfrac{1+\sin\theta}{1+\cos\theta}\right)$ **38.** $\dfrac{\tan\theta+1}{\tan\theta-1}=\dfrac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$

39. $\dfrac{\tan^2\theta+\sec^2\theta}{\sec^2\theta}=\sin^2\theta+1$ **40.** $\dfrac{\cot^2\theta+\csc^2\theta}{\csc^2\theta}=1+\cos^2\theta$ **41.** $\dfrac{1+\cot^2\theta}{\tan^2\theta}=\cot^2\theta\csc^2\theta$

42. $\dfrac{1}{\cot\theta+\tan\theta}=\sin\theta\cos\theta$ **43.** $\dfrac{1}{\sec\theta-\cos\theta}=\cot\theta\csc\theta$ **44.** $\dfrac{\cot\theta}{\sec\theta}=\csc\theta-\sin\theta$

45. $\dfrac{\cot^2\theta}{\csc\theta+1}=\csc\theta-1$ **46.** $\dfrac{\tan\theta}{\csc\theta}=\sin\theta\tan\theta$ **47.** $\dfrac{\tan^2\theta}{\sec\theta-1}=1+\sec\theta$

48. $\tan y\sin y=\sec y-\cos y$ **49.** $2\cos^2 x-1=\cos^2 x-\sin^2 x$ **50.** $\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$

51. $\dfrac{1+\sin y}{\cos y}=\dfrac{\cos y}{1-\sin y}$ **52.** $\dfrac{1}{\sec x-\tan x}=\sec x+\tan x$ **53.** $\dfrac{\cot x+1}{\cot x-1}=\dfrac{\sin x+\cos x}{\cos x-\sin x}$

54. $\dfrac{1+\sin y}{1-\sin y}=\dfrac{\csc y+1}{\csc y-1}$ **55.** $\dfrac{\cos x}{\sec x-\tan x}=\dfrac{\cos^2 x}{1-\sin x}$ **56.** $\dfrac{\cos x}{\cos x+\sin x}=\dfrac{\cot x}{1+\cot x}$

57. $\dfrac{1}{1-\sin x}+\dfrac{1}{1+\sin x}=2\sec^2 x$ **58.** $\dfrac{\cos y}{\csc y+1}+\dfrac{\cos y}{\csc y-1}=2\tan y$ **59.** $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$

60. $\cot^2 x-\cos^2 x=\cot^2 x\cos^2 x$ **61.** $\csc^2 y+\sec^2 y=\sec^2 y\csc^2 y$ **62.** $\dfrac{\tan x-\cot x}{\tan x+\cot x}=\dfrac{\tan^2 x-1}{\sec^2 x}$

63. $\dfrac{\cot^2 x-1}{\cot^2 x+1}=\cos^2 x-\sin^2 x$ **64.** $\dfrac{1-\sin x}{1+\sin x}=(\tan x-\sec x)^2$ **65.** $\sec^4 x-\sec^2 x=\tan^4 x+\tan^2 x$

66. $\tan x-\cot x=\dfrac{\sin^2 x-\cos^2 x}{\sin x\cos x}$

In problems 67–76 show by counter example that each equation is not an identity.

67. $\sin\theta = 1 - \cos\theta$

68. $\tan^2\theta - \cot^2\theta = 1$

69. $\sec\theta = \dfrac{1}{\csc\theta}$

70. $\sin\theta = \dfrac{1}{\cos\theta}$

71. $\sin^2\theta - 2\cos\theta\sin\theta + \cos^2\theta = 2$

72. $\tan^2\theta - \tan\theta = 0$

73. $\csc\theta + \sec\theta\cot\theta = 2$

74. $\sin\theta + 2\sin\theta\cos\theta = 0$

75. $\dfrac{1-\cos\theta}{1+\cos\theta} = \sin^2\theta$

76. $\dfrac{1}{\tan\theta + \csc\theta} = \sec\theta$

77. a. Verify by calculation that $(\csc^2\theta - 1)(\sec^2\theta - 1) = 1$ for the values $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{\pi}{4}$.

b. Is this equation an identity?

78. a. Verify by calculation that $\dfrac{\sin\theta - \cos\theta}{\cos\theta} = \tan\theta - 1$ for the values $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{3\pi}{4}$.

b. Is this equation an identity?

79. a. Verify by calculation that $2\sin^2\theta + \sin\theta = 1$ for the values $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{3\pi}{2}$.

b. Is this equation an identity?

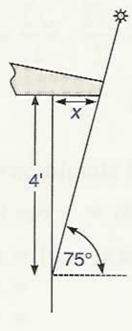
80. a. Verify by calculation that $\tan^4\theta - \tan^2\theta = 6$ for the values $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{4\pi}{3}$.

b. Is this equation an identity?

Skill and review

1. The point $(6,-2)$ is on the terminal side of θ, an angle in standard position. Compute $\sin\theta$, $\cos\theta$, and $\tan\theta$.

2. Find the length of overhang x required to shade 4 feet down the side of a house when the sun is 75° above the horizon, as shown in the figure.



3. Find the degree measure of an angle of radian measure $-\dfrac{6\pi}{5}$.

4. Simplify $\cos(\tan^{-1}(-\frac{5}{2}))$.

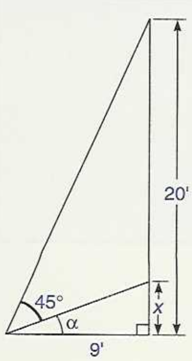
5. Use the graph of $y = |x|$ to graph the function $f(x) = |x - 3|$.

6. Graph $y = \cos\left(2x - \dfrac{\pi}{3}\right)$. State the amplitude, period, and phase shift.

7. Solve the equation $4\sin^2 x - 1 = 0$ for $0 \le x < 2\pi$. (This implies answers should be in radian measure.)

7–2 Sum and difference identities

Find the exact value of x in the figure.



In this section we describe identities that can be used to solve this problem. These identities are also useful in simplifying certain problems in higher mathematics, and they are used to derive other important identities. Some of these derivations are shown in section 7–3.

Four important identities are called the **sum and difference identities.**

Sum and difference identities for sine and cosine

[1]	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
[2]	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
[3]	$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
[4]	$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

The last three of these four identities can be developed using the first. Their verification is left as exercises. A demonstration that identity [1] is true is given in appendix A. These identities have several applications, as illustrated in example 7–2 A.

■ *Example 7–2 A*

1. Use the fact that $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$ to find the exact value of $\cos\frac{7\pi}{12}$.

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} && \text{Identity [1]}\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

This answer can and should be checked with a calculator.

2. Show that $\cos(\pi - \theta) = -\cos\theta$ for any angle θ.

$$\begin{aligned}\cos(\pi - \theta) &= \cos\pi\cos\theta + \sin\pi\sin\theta\\ &= (-1)\cos\theta + 0\sin\theta\\ &= -\cos\theta\end{aligned}$$

■

The cofunction identities

Identity [1] can be used to prove the following identities (the proofs are left for the exercises). These identities are called the **cofunction identities.**

Cofunction identities

[5]	$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	[6]	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
[7]	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	[8]	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
[9]	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$	[10]	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

The reason for the name of these identities is as follows. When the sum of two angles is 90°, or $\frac{\pi}{2}$ radians, the angles are said to be **complementary.** The angles $\frac{\pi}{2} - \theta$ and θ add up to $\frac{\pi}{2}$, so they are complementary angles. Each is said to be the complement of the other. The cofunction identities say

trig function (angle) = "co" trig function (complement of angle)

Thus, the sine and "co"sine appear in one identity, the tangent and "co"tangent appear in another, and the secant and "co" secant in the third. Whenever the sum of two angles is $\frac{\pi}{2}$ (or 90°) a trigonometric function of one equals the "cotrigonometric" function of the other. Thus for example, the following statements are true:

$$\sin 50° = \cos 40° \qquad 50° + 40° = 90°$$

$$\sec\frac{\pi}{6} = \csc\frac{\pi}{3} \qquad \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\cot 130° = \tan(-40°) \qquad 130° + (-40°) = 90°$$

■ ***Example 7–2 B***

1. Rewrite $\csc\frac{2\pi}{5}$ in terms of its cofunction.

$$\csc\frac{2\pi}{5} = \sec\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = \sec\frac{\pi}{10}$$

2. Simplify the expression: $\frac{\sin 10°}{\cos 80°}$.

$$\frac{\sin 10°}{\cos 80°} = \frac{\cos 80°}{\cos 80°} = 1$$

■

Sum and difference identities for the tangent function

Two more important identities are sum and difference formulas for the tangent function.

Sum and difference identities for tangent

[11] $$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

[12] $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

The derivation of the first identity is as follows:

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \text{Divide numerator and denominator by } \cos\alpha\cos\beta\\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{\sin\beta}{\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

Example 7–2 C illustrates an application of these identities.

■ ***Example 7–2 C***

Use the fact that $15° = 45° - 30°$ to find the exact value of $\tan 15°$.

$$\begin{aligned}\tan 15° &= \tan(45° - 30°)\\ &= \frac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°} = \frac{1-\dfrac{\sqrt{3}}{3}}{1+1\left(\dfrac{\sqrt{3}}{3}\right)}\\ &= \frac{1-\dfrac{\sqrt{3}}{3}}{1+\dfrac{\sqrt{3}}{3}}\cdot\frac{3}{3} = \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}\end{aligned}$$

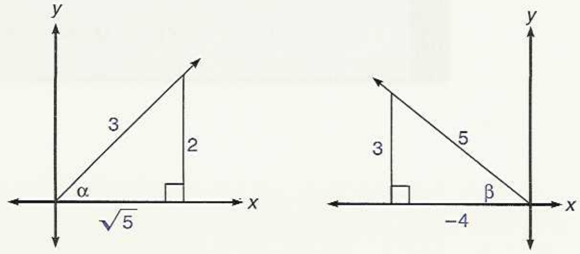
Check the result with a calculator. ■

Some problems can be solved by using the identities above and reference triangles (section 5–4). Example 7–2 D illustrates.

■ ***Example 7–2 D***

$\sin\alpha = \frac{2}{3}$, α in quadrant I; $\cos\beta = -\frac{4}{5}$, β in quadrant II. Find the exact value of $\cos(\alpha - \beta)$.

We first create reference triangles for each angle. This allows us to find any necessary trigonometric function values as necessary.



$$\begin{aligned}\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= \frac{\sqrt{5}}{3}\cdot\left(\frac{-4}{5}\right) + \frac{2}{3}\cdot\frac{3}{5}\\ &= -\frac{4\sqrt{5}}{15} + \frac{6}{15}\\ &= \frac{-4\sqrt{5}+6}{15}\end{aligned}$$

We find the necessary values from the reference triangles

Mastery points

Can you

- State the sum and difference identities?
- State and apply the cofunction identities?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for certain angles?
- Apply the sum and difference identities to find exact values of sine, cosine, and tangent for $\alpha + \beta$ and $\alpha - \beta$ given information about α and β?
- Verify identities using the sum and difference identities?

Exercise 7–2

Rewrite each function in terms of its cofunction.

1. $\sin 18°$ **2.** $\cos 42°$ **3.** $\tan 8°$ **4.** $\csc 100°$ **5.** $\sec\frac{\pi}{3}$

6. $\cot\frac{\pi}{6}$ **7.** $\cos\frac{5\pi}{6}$ **8.** $\sin\left(-\frac{\pi}{3}\right)$ **9.** $\sec\left(-\frac{3\pi}{4}\right)$ **10.** $\csc\left(-\frac{\pi}{4}\right)$

Simplify each expression.

11. $\frac{\cos 65°}{\sin 25°}$ **12.** $\tan\frac{\pi}{3}\tan\frac{\pi}{6}$ **13.** $\cos 20°\csc 70°$ **14.** $\frac{\sin^2 5°}{\cos^2 85°}$

15. $\sin\frac{\pi}{5}\sec\frac{3\pi}{10}$ **16.** $\cos^2 25° + \cos^2 65°$ **17.** $\tan^2 8° - \csc^2 82°$ **18.** $\frac{\cos^2 30°}{1-\cos^2 60°}$

19. $\tan 40°\tan 50°$ **20.** $\tan 19°\tan 71°$ **21.** $\sec\frac{\pi}{6}\sin\frac{\pi}{3}$ **22.** $\cot\frac{\pi}{5}\cot\frac{3\pi}{10}$

23. $\sin^2 10° + \sin^2 80°$ **24.** $\tan^2 25° - \csc^2 65°$ **25.** $\sec^2\frac{\pi}{3} - \cot^2\frac{\pi}{6}$ **26.** $\cos^2\frac{3\pi}{8} + \cos^2\frac{\pi}{8}$

Use the sum and difference identities to find the exact value of each of the following expressions. Observe that each value is the sum or difference of values chosen from $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°).

27. $\cos\frac{\pi}{12}$ **28.** $\tan\frac{\pi}{12}$ **29.** $\sin\frac{5\pi}{12}$ **30.** $\cos\frac{5\pi}{12}$ **31.** $\sin\frac{7\pi}{12}$ **32.** $\tan\frac{7\pi}{12}$

33. $\sin 15°$ **34.** $\tan 15°$ **35.** $\cos 105°$ **36.** $\sin 105°$ **37.** $\tan 75°$ **38.** $\cos 15°$

Each of the following problems presents information about two angles, α and β, including the quadrant in which the angle terminates. Use the information to find the required value.

39. $\cos\alpha = \frac{1}{3}$, quadrant I; $\sin\beta = \frac{3}{4}$, quadrant I. Find $\sin(\alpha+\beta)$.

40. $\cos\alpha = -\frac{12}{13}$, quadrant II; $\sin\beta = \frac{1}{2}$, quadrant II. Find $\cos(\alpha-\beta)$.

41. $\sin\alpha = \frac{5}{13}$, quadrant II; $\cos\beta = -\frac{3}{4}$, quadrant III. Find $\tan(\alpha-\beta)$.

42. $\sin\alpha = -\frac{4}{5}$, quadrant IV; $\sin\beta = -\frac{1}{5}$, quadrant IV. Find $\sin(\alpha+\beta)$.

43. $\sin\alpha = -\frac{4}{5}$, quadrant IV; $\cos\beta = \frac{15}{17}$, quadrant IV. Find $\cos(\alpha+\beta)$.

44. $\cos\alpha = -\frac{3}{5}$, quadrant II; $\sin\beta = -\frac{8}{17}$, quadrant III. Find $\sin(\alpha-\beta)$.

45. $\sin\alpha = \frac{2}{3}$, quadrant I; $\cos\beta = -\frac{1}{3}$, quadrant III. Find $\cos(\alpha-\beta)$.

46. $\cos\alpha = \frac{\sqrt{2}}{2}$, quadrant IV; $\sin\beta = -\frac{\sqrt{3}}{2}$, quadrant III. Find $\tan(\alpha+\beta)$.

47. $\sin\alpha = \frac{2}{3}$, quadrant I; $\tan\beta = \frac{1}{4}$, quadrant I. Find $\sin(\alpha+\beta)$.

48. $\tan\alpha = \frac{3}{4}$, quadrant III; $\sin\beta = -\frac{4}{5}$, quadrant III. Find $\cos(\alpha-\beta)$.

49. $\cos\alpha = \frac{5}{13}$, quadrant IV; $\tan\beta = -\frac{5}{12}$, quadrant IV. Find $\tan(\alpha-\beta)$.

50. $\cos\alpha = \frac{1}{2}$, quadrant I; $\cos\beta = \frac{\sqrt{2}}{2}$, quadrant IV. Find $\sin(\alpha+\beta)$.

51. $\tan\alpha = 2$, quadrant III; $\cos\beta = -\frac{3}{5}$, quadrant II. Find $\cos(\alpha-\beta)$.

52. $\sin\alpha = -\frac{15}{17}$, quadrant III; $\tan\beta = -\frac{3}{4}$, quadrant IV. Find $\tan(\alpha+\beta)$.

53. $\sin\alpha = \frac{2}{3}$, quadrant I; $\sin\beta = -\frac{1}{5}$, quadrant III. Find $\sin(\alpha-\beta)$.

54. $\cos\alpha = -\frac{5}{13}$, quadrant III; $\cos\beta = -\frac{8}{17}$, quadrant III. Find $\cos(\alpha+\beta)$.

55. $\cos\alpha = -\frac{\sqrt{2}}{2}$, quadrant II; $\sin\beta = \frac{\sqrt{3}}{2}$, quadrant II. Find $\tan(\alpha+\beta)$.

56. $\cos\alpha = -\frac{3}{5}$, quadrant III; $\sin\beta = \frac{1}{3}$, quadrant II. Find $\tan(\alpha-\beta)$.

Use the sum and difference identities to verify the following identities.

57. $\sin(\pi-\theta) = \sin\theta$ **58.** $\sin(\pi+\theta) = -\sin\theta$ **59.** $\cos(\pi-\theta) = -\cos\theta$

60. $\cos(\pi+\theta) = -\cos\theta$ **61.** $\tan(\pi-\theta) = -\tan\theta$ **62.** $\tan(\pi+\theta) = \tan\theta$

63. Use the sum formula to show that the sine function is 2π-periodic; that is that $\sin(\theta+2\pi) = \sin\theta$.

64. Use the sum formula to show that the cosine function is 2π-periodic; that is that $\cos(\theta+2\pi) = \cos\theta$.

65. Use the sum formula to show that the tangent function is π-periodic; that is that $\tan(\theta+\pi) = \tan\theta$.

The following identities are important because they express a product of factors as a sum of terms. Verify each identity.

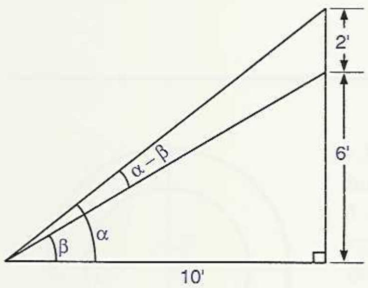
66. $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

67. $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

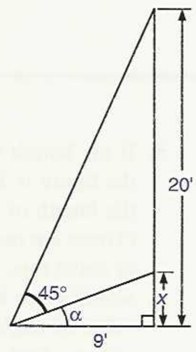
68. $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

69. $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\sin(\alpha+\beta)]$

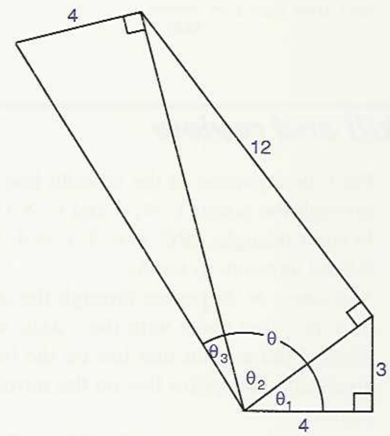
$\boxed{70.}$ A picture on a wall is 2 feet tall and 6 feet above eye level; see the diagram. Compute the exact value of $\tan(\alpha - \beta)$.



71. Referring to the figure, find the value of $\tan \alpha$ and use this to find the exact value of x. *Hint:* Compute $\tan(\alpha + 45°)$.



72. Use the identities for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ to find $\sin \theta$ in the diagram.



The following problems are designed to show that the sum and difference identities for sine and cosine [2], [3], and [4] and the cofunction identities [5] through [10] are true, using the fact that identity [1] is true. The problems are in the necessary logical order.

73. Use the identity [1] $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ to verify the identity [2] $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. Do this by replacing β by $(-\beta)$ in the identity for $\cos(\alpha + \beta)$ and simplifying, using the even and odd properties for the sine and cosine functions.

74. Verify the identity [5] $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$ by using identity [2], letting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$.

75. Identity [6] $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ is really the same as identity [5]. This can be shown as follows. Let $\alpha = \frac{\pi}{2} - \theta$, so that $\theta = \frac{\pi}{2} - \alpha$. Replace $\frac{\pi}{2} - \theta$ by α in the left member of identity [5], then replace θ in the right member by $\frac{\pi}{2} - \alpha$. Then observe that α and θ are arbitrary values, so the result can be rewritten in terms of θ.

$\boxed{76.}$ Verify identity [3] $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ as follows.

$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right) \quad \text{This is identity [5], which we know is true}$$

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] \quad \text{Replace } \theta \text{ by } \alpha + \beta$$

$$\sin(\alpha + \beta) = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] \quad \text{Regroup } \frac{\pi}{2} - \alpha - \beta$$

Now use identity [2] to expand the right member of this equation, then apply identities [5] and [6] to simplify the result and obtain identity [3].

77. Use identity [3] $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ to verify identity [4] $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$. Do this by replacing β by $(-\beta)$ in identity [3].

78. Verify identity [7] $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$ by using the fact that $\tan x = \frac{\sin x}{\cos x}$.

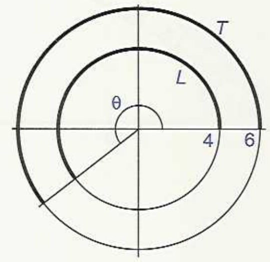
79. Verify identity [8] $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$. See problem 78 for guidance.

80. Verify identity [9] $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$ by using the fact that $\sec x = \frac{1}{\cos x}$.

81. Verify identity [10] $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$. See problem 80 for guidance.

Skill and review

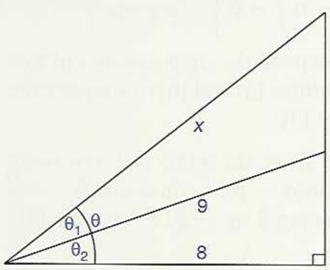
1. Find the equation of the straight line that passes through the points $(-4,3)$ and $(-8,11)$.
2. In right triangle ABC, $a = 3$, $c = 4$. Solve the triangle. Round answers to tenths.
3. The line $y = 3x$ passes through the origin. What angle does this line make with the x-axis, to the nearest 0.1°? (*Hint:* Find a point that lies on the line in the first quadrant. This point lies on the terminal side of the angle.)
4. Factor $3x^4 - 5x^3 - 14x^2 + 20x + 8$. Use the rational zero theorem and synthetic division as necessary.
5. If the length of arc L in the figure is 14.6, find the length of the arc T. (There are many ways to solve this. One way would be to find the angle θ, in value of radians, first. Recall the formula $L = rs$.)



6. Simplify $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$.
7. Verify the identity $\frac{\csc^2 x - 1}{\sin^2 x} = \cos^2 x \csc^4 x$.

7–3 The double-angle and half-angle identities

In the figure, $\theta_1 = \theta_2$. Find the length of the side marked x.



In this section we investigate identities which could be used to solve this problem. They find wider application in advanced mathematics.

Double-angle identities

Some more important identities are the **double-angle identities.** Recall that if we multiply a value by two we say we "double" the value.

Double-angle identities

[1]	$\sin 2\alpha = 2\sin\alpha\cos\alpha$
[2-a]	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
[2-b]	$\cos 2\alpha = 1 - 2\sin^2\alpha$
[2-c]	$\cos 2\alpha = 2\cos^2\alpha - 1$
[3]	$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

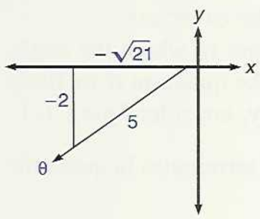
Observe that we present three identities for $\cos 2\alpha$. This is because identities [2-b] and [2-c] get so much use in the development of other identities.

The proof of identity [1] is as follows.

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$	Sum identity from section 7–2
$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$	Let $\beta = \alpha$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$	$\alpha + \alpha = 2\alpha$

The verification of the remaining identities is left for the exercises. They are done in a similar way, starting with the identities for $\cos(\alpha + \beta)$ and $\tan(\alpha + \beta)$. Example 7–3 A illustrates applying these identities.

■ ***Example 7–3 A***



1. If $\sin\theta = -\frac{2}{5}$ and θ terminates in quadrant III, find exact values of $\sin 2\theta$ and $\cos 2\theta$.

We first construct a reference triangle for θ to obtain any required trigonometric function values for that angle.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\left(-\frac{2}{5}\right)\left(-\frac{\sqrt{21}}{5}\right) = \frac{4\sqrt{21}}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{\sqrt{21}}{5}\right)^2 - \left(-\frac{2}{5}\right)^2 = \frac{21}{25} - \frac{4}{25} = \frac{17}{25}$$

2. Find an identity for $\tan 3\theta$ in terms of $\tan\theta$.

$$\tan 3\theta = \tan(2\theta + \theta) \qquad 3\theta = 2\theta + \theta$$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta} \qquad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1 - \tan^2\theta}\tan\theta} \qquad \text{Replace } \tan 2\theta \text{ by } \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2\tan\theta + \tan\theta(1 - \tan^2\theta)}{(1 - \tan^2\theta) - 2\tan^2\theta} \qquad \text{Multiply numerator and denominator by } (1 - \tan^2\theta)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad \text{Combine}$$

■

Half-angle identities

A further set of important identities is the half-angle identities.

Half-angle identities

[4] $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$

[5] $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$

[6] $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}}$

[6-a] $\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}$

[6-b] $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}$

We verify identity [5] as follows:

$$2\cos^2\theta - 1 = \cos 2\theta \qquad \text{Identity [2-c]}$$
$$2\cos^2\theta = 1 + \cos 2\theta$$
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\cos^2\frac{\alpha}{2} = \frac{1 + \cos\alpha}{2} \qquad \text{Replace } \theta \text{ by } \frac{\alpha}{2}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}} \qquad \text{Take the square root of each member}$$

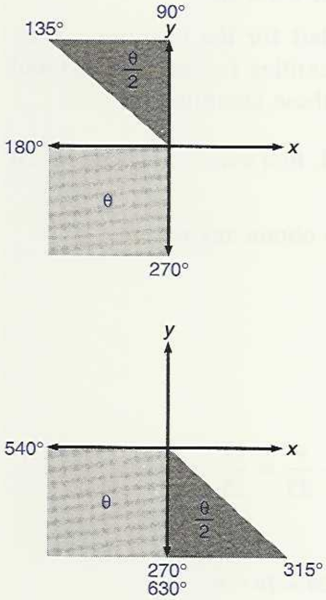


Figure 7–1

The verification of the remaining identities is left for the exercises.

The choice of plus or minus depends on the quadrant in which the angle in question terminates. It is only possible to determine the quadrant if we have information about the measure of the angle. To see why, consider figure 7–1. If $180° < \theta < 270°$ then $90° \le \frac{\theta}{2} \le 135°$; in this case, θ terminates in quadrant III and $\frac{\theta}{2}$ terminates in quadrant II. However, if $540° \le \theta \le 630°$, then $270° \le \frac{\theta}{2} \le 315°$. In this case, θ also terminates in quadrant III, but $\frac{\theta}{2}$ terminates in quadrant IV.

The half-angle identities have applications such as those shown in example 7–3 B.

■ ***Example 7–3 B***

1. Use the fact that 22.5° is one half of 45° to find the exact value of sin 22.5°.

$$\sin 22.5° = \sin\frac{45°}{2}$$
$$= \pm\sqrt{\frac{1 - \cos 45°}{2}} \qquad \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}};\ \alpha = 45°$$
$$= \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} \qquad \text{We know } \sin 22.5° > 0$$
$$= \sqrt{\frac{2 - \sqrt{2}}{4}} \qquad \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(\frac{2 - \sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2 - \sqrt{2}}}{2}$$

2. $\cos\theta = \frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find the exact value for $\cos\frac{\theta}{2}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, so $\frac{\theta}{2}$ terminates in quadrant II, where the cosine function is negative.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$$
Choose minus since $\frac{\theta}{2}$ terminates in quadrant II where $\cos\frac{\theta}{2} < 0$

$$= -\sqrt{\frac{1+\frac{3}{5}}{2}}$$
Replace $\cos\theta$ with $\frac{3}{5}$

$$= -\sqrt{\tfrac{1}{2}\cdot\tfrac{8}{5}} = -\sqrt{\tfrac{4}{5}} = -\sqrt{\tfrac{20}{25}} = -\frac{2\sqrt{5}}{5}$$ ■

It is important to understand how to rewrite identities with different forms of the argument. For example the following identities are all the same; the argument of each is shown in different forms.

[1] $\sin 2\alpha = 2\sin\alpha\cos\alpha$ — Identity [1] of the double-angle identities

$\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha$ — Replace α in identity [1] by 2α

$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ — Replace α in identity [1] by $\frac{\alpha}{2}$

Example 7–3 C illustrates rewriting identities.

■ ***Example 7–3 C***

Rewrite each expression as an expression of the form $a\sin x$, $a\cos x$, or $a\tan x$, for appropriate values of a and x.

1. $\dfrac{4\tan 2\theta}{1-\tan^2 2\theta}$

$$\frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha$$
Identity [3]

$$\frac{4\tan\alpha}{1-\tan^4\alpha} = 2\tan 2\alpha$$
Multiply each member by 2

$$\frac{4\tan 2\theta}{1-\tan^2 2\theta} = 2\tan 4\theta$$
Replace α by 2θ

Thus, the answer is $2\tan 4\theta$.

2. $\cos^2 80° - \sin^2 80°$

Compare

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$$
Identity [2] of the double-angle identities

$$\cos^2 80° - \sin^2 80°$$

Since 80° replaces α, we know that $\cos 2\alpha$ becomes $\cos 2(80)° = \cos 160°$. Thus, $\cos^2 80° - \sin^2 80° = \cos 160°$, and the answer is $\cos 160°$. ■

A similar idea is illustrated in example 7–3 D.

Example 7–3 D

Find a value of θ for which the statement $\sin 100° = 2 \sin \theta \cos \theta$ is true, then rewrite the statement replacing θ by this value.

Compare

$$\sin 110° = 2 \sin \theta \cos \theta$$
$$\sin 2\theta = 2 \sin \theta \cos \theta$$

Let $2\theta = 110°$, so $\theta = 55°$.
The statement becomes $\sin 110° = 2 \sin 55° \cos 55°$.

The identities of this and the previous sections may be combined to verify new identities.

Example 7–3 E

Verify the following identities.

1. $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2\theta}$

It is best to work with the right member since it is more complicated.

$$\frac{2 \tan \theta}{1 + \tan^2\theta} = 2 \cdot \frac{\tan \theta}{\sec^2\theta} \qquad \tan^2\alpha + 1 = \sec^2\alpha$$
$$= 2 \cdot \tan \theta \cos^2\theta \qquad \cos \alpha = \frac{1}{\sec \alpha}$$
$$= 2 \cdot \frac{\sin \theta}{\cos \theta} \cos^2\theta$$
$$= 2 \sin \theta \cos \theta \qquad \frac{1}{\cos \theta} \cdot \cos^2\theta = \cos \theta$$
$$= \sin 2\theta \qquad \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

2. $\sin 4\theta = 8 \sin \theta \cos^3\theta - 4 \sin \theta \cos \theta$

$$\sin 4\theta = \sin [2(2\theta)]$$
$$= 2 \sin 2\theta \cos 2\theta \qquad \text{Use } \sin 2\alpha = 2 \sin \alpha \cos \alpha, \text{ with } \alpha = 2\theta$$
$$= 2(2 \sin \theta \cos \theta)(2 \cos^2\theta - 1)$$
$$= 4 \sin \theta \cos \theta\, (2 \cos^2\theta - 1)$$
$$= 8 \sin \theta \cos^3\theta - 4 \sin \theta \cos \theta$$

Mastery points

Can you

- Write the double-angle and half-angle identities?
- Use the double-angle and half-angle identities to find exact values of $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$?
- Use the double-angle and half-angle identities to derive new identities and to verify given identities?
- Rewrite certain identities as a trigonometric function of $k\theta, k \in J$?

Exercise 7–3

Use the identities of this section to rewrite each expression as an expression of the form $a \sin x$, $a \cos x$, or $a \tan x$, for appropriate values of a and x.

1. $2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$ **2.** $2 \sin 52° \cos 52°$ **3.** $\cos^2 3\pi - \sin^2 3\pi$ **4.** $2 \cos^2 5\pi - 1$ **5.** $1 - 2 \sin^2 \frac{\pi}{10}$

6. $\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$ **7.** $\dfrac{6 \tan 10°}{1 - \tan^2 10°}$ **8.** $8 \cos^2 \frac{\pi}{2} - 4$ **9.** $2 \sin 6\theta \cos 6\theta$ **10.** $4 \sin 2\theta \cos 2\theta$

11. $6 \cos^2 5\theta - 3$ **12.** $8 \cos^2 3\theta - 4$ **13.** $\dfrac{10 \tan 3\theta}{1 - \tan^2 3\theta}$ **14.** $\dfrac{8 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$ **15.** $2 - 4 \sin^2 7\theta$

16. $\frac{1}{2} - \sin^2 2\theta$ **17.** $3 \cos^2 3\theta - 3 \sin^2 3\theta$ **18.** $2 \cos^2 \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2}$

Find a value of θ for which each statement is true.

19. $\sin 140° = 2 \sin \theta \cos \theta$ **20.** $\sin \theta \cos \theta = \frac{1}{2} \sin \frac{\pi}{5}$ **21.** $\cos \frac{5\pi}{6} = \cos^2\theta - \sin^2\theta$ **22.** $2 \tan 86° = \dfrac{4 \tan \theta}{1 - \tan^2\theta}$

23. $3 \cos 70° = 6 \cos^2\theta - 3$ **24.** $\cos 560° = 1 - 2 \sin^2\theta$ **25.** $\sin 10° = \sqrt{\dfrac{1 - \cos \theta}{2}}$ **26.** $\tan \theta = \sqrt{\dfrac{1 - \cos 46°}{1 + \cos 46°}}$

27. $\cos \theta = \sqrt{\frac{1}{2}\left(1 + \cos \frac{\pi}{4}\right)}$ **28.** $\sin \frac{\pi}{6} = \sqrt{\frac{1}{2}(1 - \cos \theta)}$ **29.** $\tan \frac{2\pi}{5} = \sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}}$ **30.** $\cos 40° = \sqrt{\dfrac{1 + \cos \theta}{2}}$

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for each of the following.

31. $\sin \theta = \frac{3}{5}, 0 < \theta < \frac{\pi}{2}$ **32.** $\sin \theta = -\frac{12}{13}, \pi < \theta < \frac{3\pi}{2}$ **33.** $\cos \theta = -\frac{4}{5}, \frac{\pi}{2} < \theta < \pi$

34. $\tan \theta = -\frac{3}{4}, \frac{3\pi}{2} < \theta < 2\pi$ **35.** $\csc \theta = -\frac{8}{5}, \pi < \theta < \frac{3\pi}{2}$ **36.** $\tan \theta = \frac{5}{12}, \pi < \theta < \frac{3\pi}{2}$

Find the exact value of $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ for each of the following.

37. $\sec \theta = -\frac{5}{2}, \pi < \theta < \frac{3\pi}{2}$ **38.** $\tan \theta = -\sqrt{15}, \frac{\pi}{2} < \theta < \pi$

39. $\cot \theta = -2, \frac{3\pi}{2} < \theta < 2\pi$ **40.** $\cos \theta = \frac{1}{4}, \frac{3\pi}{2} < \theta < 2\pi$

Use the half-angle identities to find the exact value of (a) $\sin \theta$, (b) $\cos \theta$, and (c) $\tan \theta$ for the following values of θ.

41. $15°$, or $\frac{\pi}{12}$ **42.** $22.5°$, or $\frac{\pi}{8}$

Use the sum/difference identities (from section 7–2) and the results of problems 41 and 42 to compute the exact value of the following. Observe that $75° = 60° + 15°$, and $37.5° = 15° + 22.5°$.

43. $\sin 75°$ **44.** $\tan 75°$ **45.** $\cos 37.5°$ **46.** $\sin 37.5°$

47. Find $\sin 7.5°$; see problem 41. **48.** Find $\cos 7.5°$; see problem 41.

Verify the following identities.

49. $\sin 2\theta + 1 = (\sin\theta + \cos\theta)^2$

50. $\cos 2\theta + 2\sin^2\theta = 1$

51. $\cos^4\theta - \sin^4\theta = \cos 2\theta$

52. $\cot\theta = \dfrac{1+\cos 2\theta}{\sin 2\theta}$

53. $\dfrac{1+\cos 2\theta}{1-\cos 2\theta} = \cot^2\theta$

54. $\tan 2\theta = \dfrac{2\tan\theta}{2-\sec^2\theta}$

55. $\cot\theta - \tan\theta = \dfrac{2\cos 2\theta}{\sin 2\theta}$

56. $2\csc 2\theta = \tan\theta + \cot\theta$

57. $\sin 2\theta - 4\sin^3\theta\cos\theta = \sin 2\theta\cos 2\theta$

58. $\cos 4\theta = 1 - 8\sin^2\theta\cos^2\theta$

59. $\csc^2\theta = \dfrac{2}{1-\cos 2\theta}$

60. $\dfrac{2\cos^3\theta}{1-\sin\theta} = 2\cos\theta + \sin 2\theta$

61. $\tan 2\theta = \dfrac{2(\tan\theta + \tan^3\theta)}{1-\tan^4\theta}$

62. $\cot 4\theta = \dfrac{1-\tan^2 2\theta}{2\tan 2\theta}$

63. $2\csc 2\theta\sin\theta\cos\theta = 1$

64. $\sec 2\theta = \dfrac{1}{1-2\sin^2\theta}$

65. $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

66. $\csc^2\dfrac{\theta}{2} = \dfrac{2}{1-\cos\theta}$

67. $\sec^2\dfrac{\theta}{2} = \dfrac{2}{1+\cos\theta}$

68. $\tan^2\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

69. $\cos^2\dfrac{\theta}{2} = \dfrac{1-\cos^2\theta}{2-2\cos\theta}$

70. $\dfrac{\csc\theta - \cot\theta}{1+\cos\theta} = \csc\theta\tan^2\dfrac{\theta}{2}$

71. $2\cos^2\dfrac{\theta}{2} - \cos\theta = 1$

72. $\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} = \dfrac{\sin\theta}{2}$

73. $\sin^2\dfrac{\theta}{2} - \cos^2\dfrac{\theta}{2} = -\cos\theta$

74. $\tan^2\dfrac{\theta}{2} + \cos^2\dfrac{\theta}{2} = \dfrac{\cos^2\theta + 3}{2+2\cos\theta}$

75. $\tan^2\dfrac{\theta}{2} = \dfrac{2}{1+\cos\theta} - 1$

76. $\sin^2\dfrac{\theta}{2} = \dfrac{\csc\theta - \cot\theta}{2\csc\theta}$

77. $4\sin^2\dfrac{\theta}{2}\cos^2\dfrac{\theta}{2} = \sin^2\theta$

78. $\dfrac{1+\sec\theta}{\sec\theta} = 2\cos^2\dfrac{\theta}{2}$

79. $\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2} = \dfrac{2}{\sin\theta}$

80. $\cot 2\theta = \dfrac{1}{2}\left(\cot\theta - \dfrac{1}{\cot\theta}\right)$ (See footnote 1.)

81. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

82. Find an identity for $\cos 3\theta$ in terms of $\cos\theta$. See the previous problem.

83. Find identities (a) for $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$ and (b) for $\cos 4\theta$ in terms of $\cos\theta$.

84. Find identities (a) for $\sin 5\theta$ in terms of $\sin\theta$ and (b) for $\cos 5\theta$ in terms of $\cos\theta$.

85. Finding the center of gravity of a certain solid involves the expression $\dfrac{3}{16}a\left(\dfrac{1-\cos 2\alpha}{1-\cos\alpha}\right)$. Show that this is equivalent to $\frac{3}{8}a(1+\cos\alpha)$.

86. Show that $\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha}$. Do this as follows. Let $\dfrac{\alpha}{2} = \theta$, so that $\alpha = 2\theta$. Replace $\dfrac{\alpha}{2}$ and α in the identity. Then simplify the right member; the most direct route will use $\cos 2\theta = 1 - 2\sin^2\theta$.

[1]See the article "A Chaotic Search for *i*," by Gilbert Strang, in *The College Mathematics Journal*, Vol. 22, No. 1, January 1991, for an interesting relationship between this identity and the subject of chaotic behavior in iterative systems.

87. Show that $\tan\dfrac{\alpha}{2} = \dfrac{\sin\alpha}{1+\cos\alpha}$. See problem 86.

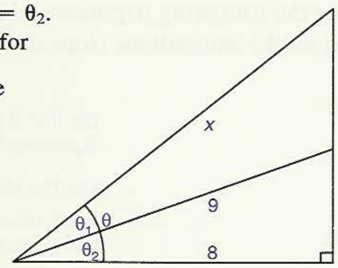
88. **a.** Use the identity for $\sin\dfrac{\theta}{2}$ with $\theta = 30°$ to find the exact value of $\sin 15°$.
b. Use $\alpha = 45°$, $\beta = 30°$ and $\sin(\alpha - \beta)$ to find the exact value of $\sin 15°$.
c. Show that the values in (a) and (b) are the same. You may find useful the principle that if $a > 0$ and $b > 0$, then $a^2 = b^2$ implies that $a = b$.

89. **a.** Find $\tan 15°$ with the identity for $\tan\dfrac{\alpha}{2}$ (half-angle identity [6]), with $\alpha = 30°$.
b. Rewrite $\tan 15°$ as $\dfrac{\sin 15°}{\cos 15°}$ and use identities [4] and [5], with $\alpha = 30°$ to compute $\tan 15°$.
c. Show that the values in parts a and b are the same.

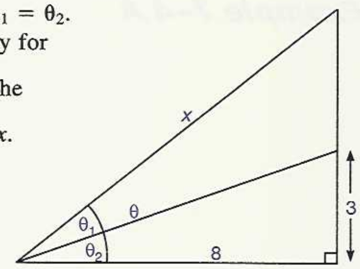
90. Verify half-angle identities [5] and [6].

91. Verify the double-angle identities [2-a], [2-b], [2-c], and [3].

92. In the figure, $\theta_1 = \theta_2$. Use the identity for $\cos\frac{\theta}{2}$ to find the length of side x.



93. In the figure, $\theta_1 = \theta_2$. Use the identity for $\tan\frac{\theta}{2}$ to find the length of side x.



The following identities are important in some situations because they relate the sums and differences of trigonometric expressions to the products of trigonometric expressions. Verify each identity.

94. $\sin 2\alpha + \sin 2\beta = 2\sin(\alpha+\beta)\cdot\cos(\alpha-\beta)$

95. $\sin 2\alpha - \sin 2\beta = 2\sin(\alpha-\beta)\cdot\cos(\alpha+\beta)$

96. $\cos 2\alpha + \cos 2\beta = 2\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)$
(*Hint:* Convert everything to cosine.)

97. $\cos 2\alpha - \cos 2\beta = -2\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)$
(*Hint:* Convert everything to cosine.)

Skill and review

1. Use the fact that $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$ and the identity for $\cos(\alpha-\beta)$ to compute $\cos\frac{\pi}{12}$.

2. Graph the rational function $f(x) = \frac{2x}{x^2 - x - 6}$.

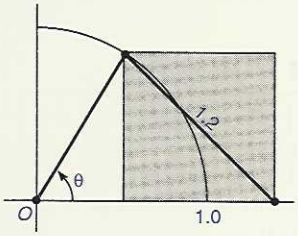
3. Find the point of intersection of the two straight lines $2x - y = 3$ and $x + 3y = 5$.

4. Rewrite $\frac{1 - \cot\theta}{\csc\theta + 1}$ in terms involving only the sine and cosine functions and simplify the result.

5. Find the least nonnegative solution to the equation $-3\sin x = 1$, to the nearest 0.1°.

7–4 Conditional trigonometric equations

Show that the area A of the shaded rectangle is given by $A = \sin\theta\,\sqrt{1.44 - \sin^2\theta}$.



This equation for area is an example of a conditional trigonometric equation. Given a value for θ we can easily compute the area A. Given a value of the area A, it would be more difficult to discover the corresponding value of θ. Conditional trigonometric equations were introduced in section 5–6. In this section we examine these equations in a more general way.

Remember that *whenever we compute an inverse trigonometric function to solve an equation we use the absolute value of the argument, which gives us the reference angle of the answer.*

Primary solutions

In this section we solve for values in both degree and radian measure. We also determine all solutions that fall between $0° \le x < 360°$ or, in radian measure, $0 \le x < 2\pi$. We call such solutions **primary solutions.** Example 7–4 A illustrates.

■ ***Example 7–4 A*** Find all primary solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $\cos\theta = -\frac{1}{2}$

$\cos\theta = -\frac{1}{2}$ — $\cos\theta < 0$ so all solutions are in quadrants II and III

$\theta' = \cos^{-1}\frac{1}{2}$ — Use the absolute value of $-\frac{1}{2}$

$\theta' = 60°$ or $\dfrac{\pi}{3}$ radians — Exact values, obtained from the unit circle, figure 6–5

$\theta = 180° \pm 60°$ or $\pi \pm \dfrac{\pi}{3}$ — In quadrant II $\theta = 180° - \theta'$; in quadrant III $\theta = 180° + \theta'$

$\theta = 120°$ or $240°$ (degrees) or $\dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$ (radians).

2. $5\sin\alpha = -2$

$\sin\alpha = -\frac{2}{5}$

Since $\sin\alpha < 0$ all solutions terminate in quadrants III and IV.

$\alpha' = \sin^{-1}\frac{2}{5}$ — As noted earlier, we use $\left|-\frac{2}{5}\right|$

$\alpha' \approx 23.6°$ or 0.4115 radians — Degree mode, radian mode

Degrees:

$$\alpha \approx 180° + 23.6° \text{ or } 360° - 23.6°$$

Radians:

$$\alpha \approx \pi + 0.4115 \text{ or } 2\pi - 0.4115$$

Thus, in degrees $\alpha \approx 203.6°$ or $336.4°$; in radians $\alpha \approx 3.5531$ or 5.8717.

3. $\tan^2 x + 4\tan x = 1$

$\tan^2 x + 4\tan x - 1 = 0$ — Quadratic, but will not factor

$\tan x = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$ — $a = 1$, $b = 4$, and $c = -1$ in $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-4 \pm \sqrt{20}}{2} = \dfrac{-4 \pm 2\sqrt{5}}{2}$ — $\sqrt{20} = \sqrt{4\cdot 5} = 2\sqrt{5}$

$= -2 \pm \sqrt{5}$

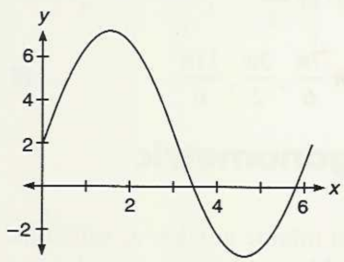
$\tan x = -2 + \sqrt{5}$	$\tan x = -2 - \sqrt{5}$
$x' = \tan^{-1}(-2 + \sqrt{5})$ $\approx 13.3°$, 0.2318 (radians) x is in quadrants I or III since $-2 + \sqrt{5} > 0$ $x \approx 13.3°$ or $180° + 13.3°$ ≈ 0.2318 or $\pi + 0.2318$	$x' = \tan^{-1}\left\|-2 - \sqrt{5}\right\|$ $= \tan^{-1}(2 + \sqrt{5})$ $\approx 76.7°$, 1.3390 (radians) x is in quadrants II or IV since $-2 - \sqrt{5} < 0$ $x \approx 180° - 76.7°$ or $360° - 76.7°$ $\approx \pi - 1.3390$ or $2\pi - 1.3390$

$x \approx 13.3°, 193.3°, 103.3°, 283.3°$ or $0.2318, 3.3734, 1.8026, 4.9442$. ■

Using a graphing calculator to solve an equation

In section 4–3, in the subsection "The TI-81 and Newton's method," we presented a way to solve equations using a graphing calculator and a built-in program called NEWTON. Example 7–4 B shows how to apply this method to part 2 of example 7–4 A.

■ ***Example 7–4 B***



Solve $5 \sin \alpha = -2$ using graphical methods.

This is equivalent to $5 \sin \alpha + 2 = 0$; find the zeros of the function $y = 5 \sin x + 2$.

Graph $y = 5 \sin x + 2$. This is shown in the figure, with Xmin = −.1, Xmax = 6.3, Ymin = −3, Ymax = 7. *The calculator is in radian mode.*

Use the trace feature to position the cursor near the zero between 3 and 4. Now execute the program NEWTON. The value 3.5531095 appears. This is one of the zeros.

Graph the function again, select trace, and position the cursor near the second zero and run the program NEWTON again. The value 5.871668461 appears. This is an approximation to the second zero.

Repeating these steps in degree mode will find approximations to the zeros in degree mode. Of course, Xmin should be something like −10°, and Xmax about 360°. ■

Using identities to help solve an equation

When an expression involves more than one trigonometric function we often use identities to rewrite the equation in terms of a single trigonometric function. This is illustrated in the next example.

■ ***Example 7–4 C***

Find all primary solutions for the following trigonometric equations. Find the solutions in degrees and radians.

1. $\tan \theta - \cot \theta = 0$

$$\tan \theta - \frac{1}{\tan \theta} = 0 \qquad \cot \theta = \frac{1}{\tan \theta} \text{ where } \tan \theta \neq 0$$

$$\tan^2\theta - 1 = 0 \qquad \text{Multiply each term by } \tan \theta$$

$$\tan^2\theta = 1$$

$$\tan \theta = \pm 1$$

When $\tan \theta = 1$, θ' is 45° or $\frac{\pi}{4}$ (see the unit circle, figure 6–4, or table 6–1), so using this fact and the ASTC rule we determine that the primary solutions in degrees are 45°, 135°, 225°, 315° and in radians are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

2. $2\cos^2 x - 3\sin x - 3 = 0$

$$2(1 - \sin^2 x) - 3\sin x - 3 = 0 \qquad \cos^2\theta = 1 - \sin^2\theta$$
$$-2\sin^2 x - 3\sin x - 1 = 0$$
$$2\sin^2 x + 3\sin x + 1 = 0$$
$$(2\sin x + 1)(\sin x + 1) = 0$$

$2\sin x + 1 = 0$	$\sin x + 1 = 0$
$\sin x = -\frac{1}{2}$	$\sin x = -1$
$x = 210°, 330°$ or $\frac{7\pi}{6}, \frac{11\pi}{6}$	$x = 270°$ or $\frac{3\pi}{2}$

The primary solutions are 210°, 270°, 330° or $\frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

Finding all solutions to a trigonometric equation

We need to take note of the fact that there are an infinite number of solutions to the equations we solved in the preceding problems. Because the trigonometric functions are periodic, the set of all solutions can be found by adding all integral values of the appropriate period (2π or π) to the solution.

This can be illustrated for part 1 of example 7–4 A, where we found that the primary solutions to $\cos\theta = -\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (in radians). However, the cosine function is 2π-periodic, which means that $\cos(\theta + 2k\pi) = \cos\theta$ for any value of θ and for integer values of k. Thus, the actual set of all radian-valued solutions for this problem is $\frac{2\pi}{3} + 2k\pi$ and $\frac{4\pi}{3} + 2k\pi$, $k \in J$. This idea is illustrated in figure 7–2 and example 7–4 D.

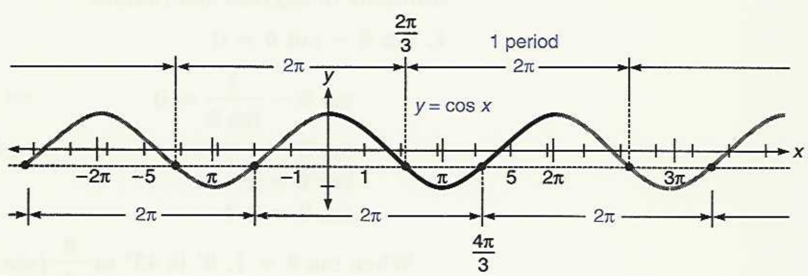


Figure 7–2

■ ***Example 7–4 D*** Find *all* solutions for the following trigonometric equations. Find the solutions in degrees (nearest tenth) and radians (four decimal places).

1. $3 \sin 2x = -2$

$\sin 2x = -\frac{2}{3}$

Since $\sin 2x < 0$, $2x$ is in quadrants III or IV.

$(2x)' = \sin^{-1} \frac{2}{3}$

$(2x)' \approx 41.8°$ or 0.7297 radians

Note 1. Do not divide both members by 2 at this point. It is necessary to find the solutions for $2x$ *before* dividing by 2.

2. Although we show the intermediate values above as 41.8° and 0.7297, it is important to keep the maximum accuracy of the calculator up to the last step of the problem. All calculators have the capability to store at least one value in memory, which should be used to avoid tedious and error-prone reentry of values.

$$2x \approx \begin{cases} 180° + 41.8° \text{ or } 360° - 41.8° \text{ (degrees)} \\ \pi + 0.7297 \text{ or } 2\pi - 0.7297 \text{ (radians)} \end{cases}$$ $2x$ is an angle in quadrants III or IV

$$2x \approx \begin{cases} 221.8°, 318.2° \text{ (degrees)} \\ 3.8713, 5.5535 \text{ (radians)} \end{cases}$$ Primary solutions for $2x$

To describe all solutions we add multiples of the period of the sine function, 360° or 2π.

$$2x \approx \begin{cases} 221.8° + k \cdot 360°, 318.2° + k \cdot 360° \\ 3.8713 + 2k\pi, 5.5535 + 2k\pi \end{cases}$$

We now divide each solution by 2.

$$x \approx \begin{cases} 110.9° + k \cdot 180°, 159.1° + k \cdot 180° \\ 1.9357 + k\pi, 2.7768 + k\pi \end{cases}$$

This describes *all* solutions to the equation. To find primary solutions for x we would compute the values above for $k = 0$ and $k = 1$. If $k = 2$, the solutions are greater than 360° (2π), and if k is negative the solutions are negative.

2. $\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$

$$\tan \frac{x}{2} = \frac{\sqrt{3}}{3}$$

$$\left(\frac{x}{2}\right)' = \tan^{-1} \frac{\sqrt{3}}{3} = 30°, \frac{\pi}{6}$$

$$\frac{x}{2} = 30°, 210°, \text{ or } \frac{\pi}{6}, \frac{7\pi}{6}$$ These are the primary solutions for $\frac{x}{2}$

The tangent function is π-periodic. Thus we add integer multiples of 180° (π) to obtain all solutions.

$$\frac{x}{2} = \begin{cases} 30° + k \cdot 180°, 210° + k \cdot 180° \text{ (degrees)} \\ \frac{\pi}{6} + k\pi, \frac{7\pi}{6} + k\pi \text{ (radians)} \end{cases}$$

This describes all solutions. However, $210° - 30° = 180°$, and similarly $\frac{7\pi}{6} - \frac{\pi}{6} = \pi$, so the solutions can be described more compactly.

$$\frac{x}{2} = 30° + k \cdot 180°, \text{ or } \frac{\pi}{6} + k\pi$$

$$x = 60° + k \cdot 360° \text{ or } \frac{\pi}{3} + 2k\pi \qquad \text{Multiply each member by 2}$$

All solutions for x are $x = 60° + k \cdot 360°$ or $\frac{\pi}{3} + 2k\pi$. ■

Equations involving multiples of the angle

If an equation mixes multiples of values with the values themselves, we can eliminate the multiple value with an appropriate identity.

■ ***Example 7–4 E***

Solve $\sin 2\theta - \sin \theta = 0$; find primary solutions.

$$2 \sin \theta \cos \theta - \sin \theta = 0 \qquad \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\sin \theta (2 \cos \theta - 1) = 0 \qquad \text{Factor}$$

$$\sin \theta = 0 \text{ or } 2 \cos \theta - 1 = 0 \qquad \text{Zero product property}$$

$\sin \theta = 0$	$2 \cos \theta - 1 = 0$
	$\cos \theta = \frac{1}{2}$
$\theta = 0°, 180°$, or $0, \pi$ (radians)	$\theta = 60°, 300°$, or $\frac{\pi}{3}, \frac{5\pi}{3}$ (radians)

The solutions are $0°, 60°, 180°, 300°$ or $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ (radians). ■

Mastery points

Can you

- Solve linear and quadratic trigonometric equations?
- Solve trigonometric equations involving multiple angles?
- Solve trigonometric equations by applying an appropriate identity?

Exercise 7–4

Find all primary solutions to the following trigonometric equations. Leave answers in both degrees and radians. All answers should be exact.

1. $\tan\theta + 1 = 0$
2. $\sin\theta - 1 = 0$
3. $2\cos\theta - 1 = 0$
4. $2\cos\theta + 1 = 0$
5. $\sqrt{3}\tan\theta - 1 = 0$
6. $\cot\theta + \sqrt{3} = 0$
7. $\csc\theta + 2 = 0$
8. $\sec\theta - 2 = 0$
9. $3\sin^2\theta - 3 = 0$
10. $3\csc^2\theta = 3$
11. $\sec^2\theta = 1$
12. $\tan^2\theta - 1 = 0$
13. $(\cos\theta - 1)(\sin\theta + 1) = 0$
14. $(\sec\theta + 2)(\csc\theta - 2) = 0$
15. $(2\cos^2\theta - 1)(\cot\theta - 1) = 0$
16. $(3\tan^2\theta - 1)(\sqrt{3}\sec\theta - 2) = 0$
17. $\sin^2\theta - \sin\theta = 0$
18. $\cos^2\theta + \cos\theta = 0$
19. $\tan^2\theta - \sqrt{3}\tan\theta = 0$
20. $\cos^2\theta - \frac{1}{2}\cos\theta = 0$
21. $2\sin^2\theta + \sin\theta - 1 = 0$
22. $\cos^2\theta + 2\cos\theta + 1 = 0$
23. $2\sin^3\theta - \sin\theta = 0$
24. $2\cos^2\theta + 3\cos\theta = 2$
25. $2\sin\theta\cos\theta - \sin\theta = 0$
26. $2\sin\theta\cos\theta + \cos\theta = 0$
27. $\sqrt{3}\tan\theta\cot\theta + \cot\theta = 0$
28. $2\tan^2\theta\cos\theta - \tan^2\theta = 0$
29. $\tan x\cot x = 0$
30. $\sin x\cos x = 0$
31. $2\sin x - \csc x + 1 = 0$
32. $2\cos x + \sec x - 3 = 0$
33. $\tan x + \cot x = -2$
34. $2 - \sin x - \csc x = 0$
35. $2\sin^2 x - \cos x = 1$
36. $2\cos^2 x - 3\sin x = 3$
37. $4\tan^2 x = 3\sec^2 x$
38. $4\cot^2 x - 3\csc^2 x = 0$
39. $\sin^2 x - \cos^2 x = 0$
40. $\cot^2 x + \csc^2 x = 0$
41. $2\tan^2 x\sin x = \tan^2 x$
42. $\sin^2 x\cos x - \cos x = 0$

Solve the following equations using the quadratic formula if necessary and a calculator. Find the primary solutions in both radians and degrees. Round radian answers to hundredths and degree answers to tenths.

43. $6\sin^2 x - 2\sin x - 1 = 0$
44. $3\cos^2 x + \cos x - 2 = 0$
45. $\cot^2 x - 3\cot x - 2 = 0$
46. $\tan^2 x + 5\tan x + 2 = 0$
47. $\sec^2 x - 2\sec x - 4 = 0$
48. $2\csc^2 x - \csc x - 5 = 0$
49. $\tan x + 2\sec x = 3$
50. $3\cot x - \csc x - 1 = 0$

Find all solutions to the following trigonometric equations in both degrees and in radians.

51. $\cos x = \frac{1}{2}$
52. $\sin x = 1$
53. $\cot x = -\sqrt{3}$
54. $\cos x = -\frac{\sqrt{3}}{2}$
55. $\sin x = -\frac{\sqrt{2}}{2}$
56. $\tan x = -1$
57. $\tan x = 1$
58. $\sec x = \frac{2}{\sqrt{3}}$
59. $\csc x = 2$
60. $\tan\frac{x}{2} = 1$
61. $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$
62. $\sin 3x = 0$
63. $\cos 3x = -1$
64. $\sec\frac{x}{2} = 1$
65. $3\cot 2x = \sqrt{3}$
66. $2\sin 3x = -1$
67. $2\cos 4x = -1$
68. $-\sqrt{3}\tan 5x = 1$
69. $2\cos 2x + 1 = 0$
70. $\tan 2\theta - 1 = 0$
71. $\cot 2\theta - \sqrt{3} = 0$
72. $2\cos 3\theta = -1$
73. $2\sin 2\theta = 1$
74. $\sin\frac{\theta}{3} = \frac{\sqrt{3}}{2}$
75. $\sec 3\theta = 2$
76. $\csc 2\theta = -\frac{2\sqrt{3}}{3}$
77. $\sqrt{3}\tan\frac{\theta}{4} = 1$
78. $\cot\frac{\theta}{3} = \frac{\sqrt{3}}{3}$

Find the primary solutions to the following trigonometric equations in both radians and degrees. Find solutions in radians to hundredths and in degrees to the nearest tenth of a degree, where necessary.

79. $\cos 2\theta + \sin\theta = 0$
80. $\cos 2\theta - \cos\theta = 0$
81. $\sin 2\theta + \sin\theta = 0$
82. $\cos^2\theta - \sin^2\theta = 1$
83. $\cos 2\theta = 1 - \sin\theta$
84. $\cos 2\theta = \cos\theta - 1$
85. $\sin\frac{\theta}{2} = \tan\frac{\theta}{2}$
86. $\sin\frac{\theta}{2} = \cos\theta$
87. $2\sec\theta = \csc^2\frac{\theta}{2}$
88. $\sin^2\frac{\theta}{2} = \cos\theta$
89. $\tan\frac{\theta}{2} = \cos\theta - 1$
90. $\cot\theta - \tan\frac{\theta}{2} = 0$
91. $\sin 2\theta - \cos\theta = \cos^2\theta$

In the mathematical modeling of an aerodynamics problem the following equation arises:

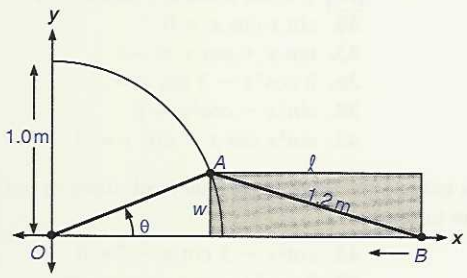
$$y = x\cos A\cos B - x^2\cos A\sin B - x^3\sin A$$

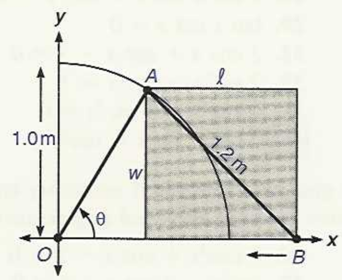
Problems 92 and 93 use this equation.

$\boxed{92.}$ If $A = 0.855$, $B = 1.052$, and $y = 0$, solve for x to the nearest 0.01.

$\boxed{93.}$ If $B = 0.7$, $x = 2$, and $y = -8$, find A to the nearest 0.01. Find the least nonnegative solution(s). (*Hint:* $\sin^2 A = 1 - \cos^2 A$.)

A mechanical device is constructed as shown in the diagram. The arm OA moves through angle θ, from 0° to 90°. Two positions are shown. Point A moves along a circle of radius 1.0 meters, and point B moves horizontally only. The distance AB is fixed by arm AB at 1.2 meters. The area of the shaded rectangle is the product of its length and width, $A = \ell w$.





94. Show that $A = \sin\theta\sqrt{1.44 - \sin^2\theta}$. (The units are square meters.)

95. Find A, to the nearest 0.01 m², when $\theta = 0°, 30°, 45°, 60°, 90°$.

$\boxed{96.}$ Find θ when $A = 0.5$ m². Round the answer to the nearest 0.1°.

Skill and review

1. Graph the function $y = -2\sin 4x$.
2. Graph the quadratic function $f(x) = x^2 + 4x - 8$ by completing the square and putting it in vertex form.
3. Use a reference triangle to find $\tan\theta$ when $\sin\theta = \frac{\sqrt{3}}{6}$ and $\cos\theta < 0$.
4. Factor $-5a^8 + 5a^2x^6$.
5. Find the equation of the straight line that passes through the point (1,3) and has x-intercept at 4.
6. Perform the indicated subtraction to collect $\frac{3}{x^2-1} - \frac{1}{x-1}$ into one term, in simplest form.

Chapter 7 summary

- **Reciprocal identities**

$$\csc\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta},\quad \cos\theta = \frac{1}{\sec\theta},\quad \tan\theta = \frac{1}{\cot\theta}$$

- **Tangent and cotangent identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- **Fundamental identity of trigonometry**

$$\sin^2\theta + \cos^2\theta = 1$$

- **Pythagorean identities**

Pythagorean identities	**Useful forms**
$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ $\cos^2\theta = 1 - \sin^2\theta$
$\sec^2\theta = \tan^2\theta + 1$	$\tan^2\theta = \sec^2\theta - 1$ $\sec^2\theta - \tan^2\theta = 1$
$\csc^2\theta = \cot^2\theta + 1$	$\cot^2\theta = \csc^2\theta - 1$ $\csc^2\theta - \cot^2\theta = 1$

- **Sum and difference identities for sine and cosine**

[1] $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

[2] $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

[3] $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

[4] $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

- When the sum of two angles is 90°, or $\frac{\pi}{2}$ radians, the angles are said to be **complementary.**

- **Cofunction identities**

[5] $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ [6] $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$

[7] $\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$ [8] $\cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$

[9] $\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta$ [10] $\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta$

- **Sum and difference identities for tangent**

[11] $\tan(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

[12] $\tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

- **Double-angle identities**

[1] $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

[2-a] $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

[2-b] $\cos 2\alpha = 1 - 2 \sin^2\alpha$

[2-c] $\cos 2\alpha = 2 \cos^2\alpha - 1$

[3] $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2\alpha}$

- **Half-angle identities**

[4] $\sin \dfrac{\alpha}{2} = \pm \sqrt{\dfrac{1 - \cos \alpha}{2}}$

[5] $\cos \dfrac{\alpha}{2} = \pm \sqrt{\dfrac{1 + \cos \alpha}{2}}$

[6] $\tan \dfrac{\alpha}{2} = \pm \sqrt{\dfrac{1 - \cos \alpha}{1 + \cos \alpha}}$

[6-a] $\tan \dfrac{\alpha}{2} = \dfrac{1 - \cos \alpha}{\sin \alpha}$

[6-b] $\tan \dfrac{\alpha}{2} = \dfrac{\sin \alpha}{1 + \cos \alpha}$

Chapter 7 review

[7–1] Verify the following identities.

1. $\dfrac{\cot \theta}{\cos \theta} = \csc \theta$
2. $\sec \theta \tan \theta = \sec^2\theta \sin \theta$
3. $\dfrac{\tan^2\theta}{\sec^2\theta - 1} = \sin^4\theta \csc^4\theta$
4. $\dfrac{\csc^2\theta - 1}{\sec^2\theta - 1} = \cot^4\theta$
5. $\dfrac{\csc \theta \tan \theta}{\sin \theta} = \csc \theta \sec \theta$
6. $\sin^2\theta - \cos^2\theta = 2 \sin^2\theta - 1$

Obtain an equivalent expression involving only the sine and cosine functions. Simplify the resulting expression.

7. $\csc \theta - \sec \theta$
8. $\tan \theta + \cot \theta$
9. $\dfrac{\sec \theta}{\tan \theta - \cot \theta}$
10. $\dfrac{1 - \cot \theta}{\csc \theta + 1}$
11. $\dfrac{\sec^2\theta - 1}{\sec^2\theta}$
12. $\dfrac{1 - \cot^2\theta}{\csc^2\theta - 1} \cos^2\theta$

Verify that each equation is an identity.

13. $\csc x - \tan x \cot x = \csc x - 1$
14. $\sin^2x + \sin^2x \cot^2x = 1$
15. $\csc^2x - \sec^2x = \csc^2x \sec^2x(\cos^2x - \sin^2x)$
16. $\dfrac{1}{\csc x - \cot x} = \dfrac{\sin x}{1 - \cos x}$
17. $(\tan x - 1)(\csc^2x - \cot^2x) = \sec x(\sin x - \cos x)$
18. $\dfrac{\csc^2x - 1}{\sin^2x} = \cos^2x \csc^4x$
19. $\dfrac{1}{1 + \csc x} + \dfrac{1}{1 - \csc x} = -2 \tan^2x$
20. $\sin^2x + \sin^2x \cos^2x = 1 - \cos^4x$
21. $\tan^4x + \tan^2x = \dfrac{\sec^2x}{\cot^2x}$
22. $\dfrac{1 - \cot x}{1 + \csc x} = \dfrac{\sin x - \cos x}{\sin x + 1}$

[7–2] Use the sum and difference identities to find the exact value of each of the following. Observe that each value is the sum or difference of values chosen from $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°).

23. $\sin \dfrac{\pi}{12}$

24. $\tan\left(-\dfrac{\pi}{12}\right)$

25. $\sin 75°$

26. $\cos(-15°)$

Each of the following problems presents information about two angles, α and β, including the quadrant in which the angle terminates. Use the information to find the required value.

27. $\cos \alpha = \frac{3}{5}$, quadrant I; $\sin \beta = \frac{5}{13}$, quadrant I. Find $\sin(\alpha + \beta)$.

28. $\cos \alpha = -\frac{12}{13}$, quadrant III; $\sin \beta = \frac{2}{3}$, quadrant I. Find $\cos(\alpha - \beta)$.

29. $\tan \alpha = -\frac{5}{12}$, quadrant II; $\sec \beta = -3$, quadrant II. Find $\tan(\alpha - \beta)$.

Use the sum and difference identities to verify the following identities.

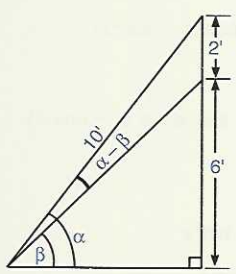
30. $\cos\left(\dfrac{3\pi}{2} + \theta\right) = \sin \theta$

31. $\sin\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\sqrt{2}}{2}(\cos \theta - \sin \theta)$

32. $\dfrac{\cos(\alpha + \beta)}{\sin \alpha \cos \beta} = \cot \alpha - \tan \beta$

33. Use the sum formula to show that the sine function is 2π-periodic; that is that $\sin(\theta + 2\pi) = \sin \theta$.

34. A picture on a wall is 2 feet tall and 6 feet above eye level (see the diagram). Compute the exact value of $\sin(\alpha - \beta)$.



[7–3] Using the double-angle identities, find the angle θ that makes the following statements true.

35. $\cos \theta = \cos^2 62° - \sin^2 62°$

36. $\sin \theta = 2 \sin 5\pi \cos 5\pi$

37. $\tan \theta = \dfrac{2 \tan \dfrac{7\pi}{12}}{1 - \tan^2 \dfrac{7\pi}{12}}$

38. $\cos 24° = 1 - 2 \sin^2\theta$

39. $a \sin b\theta = 6 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}$; find a and b.

40. Given $\cos \theta = -\frac{5}{12}$, θ terminates in quadrant III. Find the exact value of (a) $\tan 2\theta$; (b) $\sin 2\theta$.

41. Given $\sin \theta = \frac{4}{5}$, θ terminates in quadrant II. Find the exact value of (a) $\cos 2\theta$; (b) $\tan 2\theta$.

42. Verify the identity $\sin 2x - \cos x = \cos x(2 \sin x - 1)$.

43. Verify the identity $1 + \cos 2x = 2 \cos^2 x$.

Using the half-angle identities, find the exact value of the following.

44. $\tan 22.5°$

45. $\cos \dfrac{\pi}{8}$

46. Given $\cos x = \dfrac{12}{13}$, $0 < x < \dfrac{\pi}{2}$. Find (a) $\sin \dfrac{x}{2}$ and (b) $\tan \dfrac{x}{2}$.

47. Given $\sin x = -\dfrac{2}{3}$, $3\pi < x < \dfrac{7\pi}{2}$. Find (a) $\cos \dfrac{x}{2}$ and (b) $\tan \dfrac{x}{2}$.

48. Verify the identity $\cot \dfrac{\theta}{2} = \dfrac{1 + \cos \theta}{\sin \theta}$.

49. Verify the identity $\sec^2\dfrac{\theta}{2} - \tan^2\dfrac{\theta}{2} = 1$.

[7–4] Solve the following conditional equations. Find all primary solutions in both radians and degrees.

50. $2 \sin x - 1 = 0$

51. $3 \cot^2 x - 1 = 0$

52. $(\sin x - 1)(2 \cos x - 1) = 0$

53. $(4 \sin^2 x - 1)(\sec x - 2) = 0$

54. $\cot^2\theta - \cot \theta = 0$

55. $\sec^2\theta - 4 = 0$

56. $2 \cos^2\theta - \cos \theta - 1 = 0$

57. $2 \sin \theta - \csc \theta + 1 = 0$

58. $2 \cot \theta \cos \theta = \cot^2\theta$

59. $2 \sin^2\theta - 3 \cos \theta = 3$

60. $2 \sin 4x = 1$

61. $2 \cos \frac{x}{2} - \sqrt{3} = 0$

62. $\sqrt{3} \tan \frac{x}{2} + 1 = 0$

63. $\sin^2 \frac{x}{4} = \frac{1}{2}$

64. $3 \tan^2 \frac{\theta}{4} = 9$

65. $2 \cos 5\theta = 1$

66. $\cos \theta + \sin 2\theta = 0$

Solve the following conditional equations. Find all primary solutions in radians only. Round solutions to hundredths.

67. $3 \cos^2 x - 1 = 0$

68. $\sin x - 2 \csc x = 5$

69. $\cos \frac{x}{2} - \sin x = 0$

70. $\left(\tan \frac{x}{4} - \frac{\sqrt{3}}{3}\right)\left(\sin 3x + \frac{1}{2}\right) = 0$

71. $3 \sin^2 2x - \sin 2x - 2 = 0$

Chapter 7 test

1. Show that the expression $\csc^2 x \sin x \cos x$ is equivalent to $\cot x$.

2. Write the expression $\dfrac{\csc x - \sec x}{\tan x + \cot x}$ as an expression in sine and cosine, and simplify.

3. Show by counter example that the equation $\cot x - 2 \tan x = 1$ is not an identity.

4. $a \cos b\theta = 8 \cos^2 2\theta - 8 \sin^2 2\theta$; find a and b.

5. Given $\cos \alpha = -\frac{1}{2}$, α terminates in quadrant III, and $\sin \beta = \frac{15}{17}$, β terminates in quadrant II. Find $\sin(\alpha + \beta)$.

6. Given $\cos x = -\frac{8}{17}$, x terminates in quadrant II. Find $\sin 2x$.

7. Given $\csc x = \frac{5}{3}$, $\frac{5\pi}{2} < x < 3\pi$. Find $\tan \frac{x}{2}$.

8. Use the appropriate half-angle formula to find $\sec 22.5°$.

Verify the following identities.

9. $\dfrac{1 + \cot \theta}{\csc \theta} = \sin \theta + \cos \theta$

10. $\dfrac{\cos^2 x - 1}{\sin^2 x} = -1$

11. $\tan^4 x + \tan^2 x = \dfrac{\sec^2 x}{\cot^2 x}$

12. $\cos\left(\theta - \frac{3\pi}{2}\right) = -\sin \theta$

13. $\cos 2x - \sin 2x = 2 \cos x(\cos x - \sin x) - 1$

14. Find all primary solutions to the equation $4 \cos^2 x - 1 = 0$ (radians and degrees).

15. Find all primary solutions to the equation $(\cot \theta - \sqrt{3})(\sec \theta + 2) = 0$ (radians and degrees).

16. Find all primary solutions to the equation $6 \sin^2 x + 5 \sin x - 1 = 0$ (radians only). Round solutions to the nearest 0.1.

17. Find all primary solutions to the equation $\sin^2 3x = 0.5$ (radians only). Round answers to the nearest 0.1.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